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Geometry of the De-Sitter Universe

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Abstract

By making use of the fact that the de-Sitter metric corresponds to a hyperquadric in a five-dimensional flat space, it is shown that the three Robertson–Walker metrics for empty spacetime and positive cosmological constant, corresponding to 3-space of positive, negative and zero curvative, are geometrically equivalent. The 3-spaces correspond to intersections of the hyperquadric by hyperplanes, and the time-like geodesics perpendicular to them correspond to intersections by planes, in all three cases.

1. *Introduction*

The purpose of this work is to investigate the geometrical properties of the Robertson–Walker type solutions of

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.1}$$

In Einstein's theory these solutions correspond to empty universes and are devoid of physical interest. If the real universe were such that $|\Lambda| \gg \kappa\rho c^2$ ($\kappa = 8\pi G/c^4$) (and this inequality seems highly unlikely on the basis of the observational data), an empty space solution (with matter treated as test particles) could be a good approximation to the actual geometry of the universe. A more plausible physical justification for the study of cosmological solutions of (1.1) comes from Hoyle–Narlikar creation-field cosmology (Hoyle, 1963); if the density is required to remain constant this theory reduces to (1.1) with $\Lambda = \kappa\rho c^2/2$. This interpretation will be discussed more fully in a subsequent paper. In any case the geometrical aspects of (1.1) are sufficiently interesting to warrant attention apart from their possible relevance to physics.

The Robertson–Walker metrics are of the form

$$dt^2 - S^2(t)\left(1 + \frac{k\bar{r}^2}{4}\right)^{-2} (d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2)) \tag{1.2}$$

and equation (1.1) reduces to

$$\dot{S}^2 = -k + \Lambda S^2/3 \tag{1.3}$$

The curves of constant $(\bar{r}, \theta, \varphi)$ are time-like geodesics orthogonal to the space-like hypersurfaces of constant t. These space-like hypersurfaces have positive, negative or zero curvature according as $k = \pm 1$ or zero.

The solutions of (1.3) are

(a) Λ negative. Choose units so that $\Lambda = -3$

$$k = -1; \qquad S = \cos t \tag{1.4}$$

(b) Λ positive. Choose units so that $\Lambda = 3$.

$$\begin{cases} k = +1; & S = \cosh t & (1.5) \\ k = 0; & S = \exp t & (1.6) \\ k = -1; & S = \sinh t & (1.7) \end{cases}$$

Make the following change of radial variable:

$$k = +1; \qquad \tan(\rho/2) = \bar{r}/2 \tag{1.8}$$

$$k = 0; \qquad \rho = \bar{r} \tag{1.9}$$

$$k = -1; \qquad \tanh(\rho/2) = \bar{r}/2 \tag{1.10}$$

and we obtain the solutions (1.2) in the form

(a) $\dfrac{\Lambda = -3}{k = -1}$; $\quad dt^2 - \cos^2 t(d\rho^2 + \sinh^2 \rho \, d\Omega^2) \qquad (1.11)$

(b) $\Lambda = 3$

$$\begin{cases} k = +1; & dt^2 - \cosh^2 t(d\rho^2 + \sin^2 \rho \, d\Omega^2) & (1.12) \\ k = 0; & dt^2 - e^{2t}(d\rho^2 + \rho^2 \, d\Omega^2) & (1.13) \\ k = -1; & dt^2 - \sinh^2 t(d\rho^2 + \sinh^2 \rho \, d\Omega^2) & (1.14) \end{cases}$$

The metric (1.13) is the well-known de-Sitter metric. It is, of course, equivalent to the *static* de-Sitter metric

$$(1 - r^2)\, d\tau^2 - \frac{dr^2}{1 - r^2} - r^2 \, d\Omega^2 \tag{1.15}$$

the coordinate transformation connecting (1.13) and (1.15) being

$$e^{2t} = e^{2\tau}(1 - r^2), \qquad \rho = r e^{-\tau} \tag{1.16}$$

As is well known, the metric (1.15) is that of the hyperquadric

$$\eta_1{}^2 + \eta_2{}^2 + \eta_3{}^2 - \eta_4{}^2 + \eta_5{}^2 = r^2 - \eta_4{}^2 + \eta_5{}^2 = 1 \tag{1.17}$$

in a flat 5-space with signature (+++−+), metric

$$d\eta_1{}^2 + d\eta_2{}^2 + d\eta_3{}^2 - d\eta_4{}^2 + d\eta_5{}^2 \tag{1.18}$$

The parameters $\eta_1, \ldots, \eta_5$ are defined by

$$\eta_1 = r\sin\theta\cos\theta, \qquad \eta_2 = r\sin\theta\sin\varphi, \qquad \eta_3 = r\cos\theta \tag{1.19}$$

$$\eta_5 \pm \eta_4 = e^{\pm\tau}(1 - r^2)^{1/2}$$

The identity (1.17) and the equality (apart from an overall minus sign) of (1.15) and (1.18) are easily verified.

The subject matter of this introduction is dealt with in greater detail in the literature. See, for instance, Adler, Bazin & Schiffer (1965), Bondi (1960), and Fock (1964).

In fact, all three metrics (1.12)–(1.14) are the metric of the hyperquadric in different coordinate systems. This equivalence is truly remarkable since in conjunction with the interpretation of the $d\rho = d\theta = d\varphi = 0$ lines as world-lines of matter the three metrics refer to completely different cosmological situations (respectively: 3-space positively curved (closed); flat; negatively curved/infinite in time with expansion preceded by contraction; infinite in time continually expanding; creaction at $t = 0$ followed by expansion). The aim of the present paper is to provide a satisfactory visual interpretation of the geometrical situation.

2. *Geometry of the Hyperquadric*

Take a section of the hyperquadric (1.17) defined by $\theta = \text{const}$, $\varphi = \text{const}$. We get a two-dimensional subspace—a quadric in a flat 3-space with coordinates (r, η_4, η_5). We actually get only half the quadric since r takes only the positive values. We include the negative values by adopting the convention that $(-r, \theta, \varphi, t)$ means $(r, \pi - \theta, \pi + \varphi, t)$. The equation of the quadric is

$$r^2 - \eta_4{}^2 + \eta_5{}^2 = 1 \tag{2.1}$$

(a hyperboloid of one sheet), and the metric of the 3-space is

$$dr^2 - d\eta_4{}^2 + d\eta_5{}^2 \tag{2.2}$$

In terms of the parameters (t, ρ) given by

$$\left.\begin{aligned} r &= \cosh t \sin\rho \\ \eta_4 &= \sinh t \\ \eta_5 &= \cosh t \sin\rho \end{aligned}\right\} \tag{2.3}$$

the metric (2.2) is

$$\cosh^2 t \,.\, d\rho^2 - dt^2 \tag{2.4}$$

Any plane through the origin of (r, η_4, η_5)-space can, by a suitable $SO(2,1)$ rotation, be taken to be one of the two planes

$$\left.\begin{array}{ll} \eta_5 = 0 & (t=0) \\ \eta_4 = 0 & (\rho = 0) \end{array}\right\} \tag{2.5}$$

or

From the metric (2.4) we readily obtain the geodesic equations

$$\left.\begin{array}{r} dt^2/ds^2 + \sinh t \cosh t\,(d\rho/ds)^2 = 0 \\ d\rho^2/ds^2 + 2\tanh t(dt/ds)\,(d\rho/ds) = 0 \end{array}\right\} \tag{2.6}$$

which are clearly satisfied by the curves $t = 0$ (ρ a linear function of s) and $\rho = 0$ (t a linear function of s), so that the curves obtained by intersection of the quadric and the planes (2.5) are geodesics. Hence the intersection of the quadric by any plane through the origin of (r, η_4, η_5)-space is a geodesic. Conversely, every geodesic on the quadric lies in some plane through the origin of (r, η_4, η_5)-space.

We are, of course, interested not so much in the quadric (2.1) as in the hyperquadric (1.17), which in terms of t and ρ has metric

$$-dt^2 + \cosh^2 t(d\rho^2 + \sin^2 \rho\, d\Omega^2) \tag{2.7}$$

(which, incidentally, demonstrates that (1.12) is a valid form for the metric of the hyperquadric, so (1.12) and (1.13) are equivalent). The geodesic equations for this metric reduce to (2.6) under the restriction to constant θ and φ. Thus *every geodesic of the quadric is also a geodesic of the hyperquadric.*

3. *Coordinate Systems on the Quadric*

Parametrise the quadric

$$r^2 - \eta_4{}^2 + \eta_5{}^2 = 1 \tag{3.1}$$

by the coordinates (r, t) where

$$t = \tanh^{-1}(\eta_4/\eta_5) \tag{3.2}$$

The coordinate curves $r =$ constant and $t =$ constant are then, respectively, the intersections of the quadric with the planes parallel to the (η_4, η_5)-plane, and the planes $\eta_4 + K\eta_5 = 0$ (see Fig. 1). The generators of the quadric are null lines. The two generators through $(r, \eta_4, \eta_5) = (\pm 1, 0, 0)$ are the intersections of the quadric with the planes $\eta_4 = \pm\eta_5$ (corresponding to $t = \pm\infty$). Thus the null generators of $(1,0,0)$ and $(-1,0,0)$ are coordinate singularities.

From (3.1) and (3.2),

$$\left.\begin{array}{l} \eta_4{}^2 = (1 - r^2)\sinh^2 t \\ \eta_5{}^2 = (1 - r^2)\cosh^2 t \end{array}\right\} \tag{3.3}$$

so that

$$d\eta_5{}^2 - d\eta_4{}^2 = \frac{r^2\,dr^2}{1-r^2} - (1-r^2)\,dt^2$$

and the metric of the quadric in terms of (r, t) is

$$\frac{dr^2}{1-r^2} - (1-r^2)\,dt^2 \tag{3.4}$$

The corresponding metric of the hyperquadric is obtained simply by restoring the $r^2\,d\Omega^2$ term, and we arrive at the static de-Sitter metric (1.15).

Thus in Fig. 1 we have a visualisation of the coordinate system we are

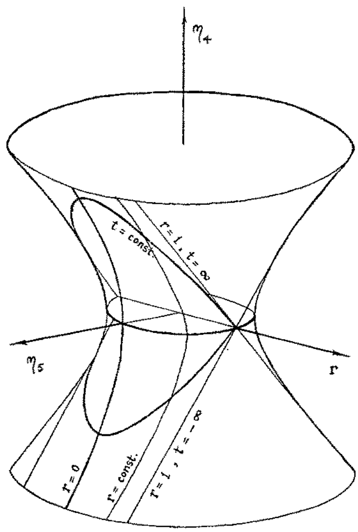

Figure 1. De-Sitter's static universe

employing when we say that the static de-Sitter metric is the metric of a hyperquadric. Note that the 4-space of the static de-Sitter metric is not the whole of the hyperquadric but only that portion bounded by null generators through the points $(\pm 1, 0, 0)$. Note that only one of the $r =$ constant curves is a geodesic ($r = 0$)—world-lines of matter in the form of 'test particles' in a *static* de-Sitter world will not be simply related to the coordinate system. In fact a static de-Sitter world cannot contain a static distribution of test particles.

If we extend the coordinate system we have set up to the regions $|r| > 1$ the $r =$ constant curves become space-like and the $t =$ constant curves become time-like. We therefore change the notation by what amounts to a

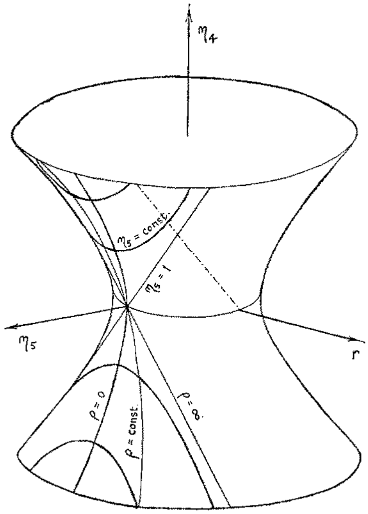

Figure 2. Universe with k = −1

rotation of the whole coordinate net through a right angle about the η_4-axis (Fig. 2). Define

$$\rho = \tanh^{-1}(r/\eta_4) \tag{3.5}$$

and express the metric in terms of (η_5, ρ). The manipulations are formally the same as those leading to (3.4), and we find that the metric is

$$-\frac{d\eta_5{}^2}{\eta_5{}^2 - 1} + (\eta_5{}^2 - 1)\, d\rho^2 \tag{3.6}$$

In terms of the new parameter t defined by

$$\eta_5 = \cosh t \tag{3.7}$$

(which does not alter the curves of Fig. 2) we get

$$-dt^2 + \sinh^2 t\, d\rho^2 \tag{3.8}$$

We have only to include $r^2 d\Omega^2 = (\eta_5{}^2 - 1)\sinh^2\rho\, d\Omega^2 = \sinh^2 t \sinh^2 \rho\, d\Omega^2$ to obtain the corresponding form for the metric of the hyperquadric:

$$-dt^2 + \sinh^2 t(d\rho^2 + \sinh^2 \rho)\, d\Omega^2 \tag{3.9}$$

Thus we have obtained (1.14) as the metric of the hyperquadric. Note that the Robertson–Walker 4-space corresponding to (1.14) corresponds only to the part of the hyperquadric for which $\eta_5 > 1$. Note that the ρ = constant curves in Fig. 2 are intersections of the quadric by planes through the origin $(r + K\eta_4 = 0)$, so are geodesics.

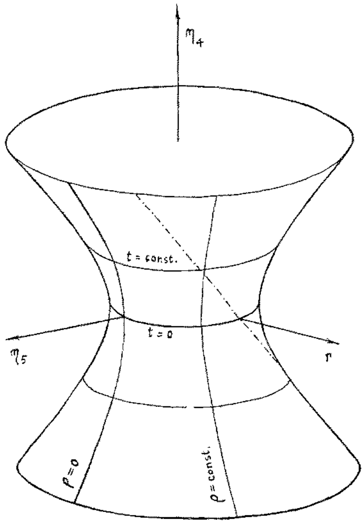

Figure 3. Universe with k = +1

The most natural coordinate systems on the quadric are obtained by taking the coordinate curves to be intersection of the quadric by planes $\eta_4 = \text{constant}$ and $\eta_5 + Kr = 0$ (i.e. through the symmetry axis of the quadric). Such a system covers the whole quadric without singularities (Fig. 3). Define

$$\rho = \tan^{-1}(r/\eta_5) \tag{3.10}$$

and use (ρ, η_4) as coordinates. Then

$$r^2 = (1 + \eta_4{}^2)\sin^2\rho$$

$$\eta_5{}^2 = (1 + \eta_4{}^2)\cos^2\rho$$

$$dr^2 + d\eta_5{}^2 = -\eta_4{}^2\, d\eta_4{}^2/(1 + \eta_4{}^2)^2 + (1 + \eta_4)^2\, d\rho^2$$

and the metric is therefore

$$-\frac{d\eta_4{}^2}{1 + \eta_4{}^2} + (1 + \eta_4{}^2)\, d\rho^2$$

In terms of the new variable t defined by

$$\eta_4 = \sinh t \tag{3.11}$$

we get

$$-\, dt^2 + \cosh^2 t \,.\, d\rho^2 \tag{3.12}$$

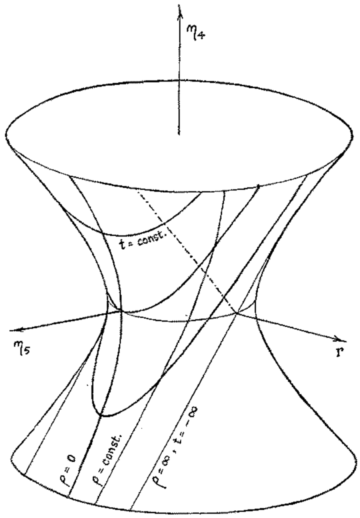

Figure 4. Universe with k = 0

For the corresponding metric of the hyperquadric we add on

$$r^2\, d\Omega^2 = (1 + \eta_4{}^2) \sin^2 \rho\, d\Omega^2 = \cosh^2 t \sin^2 \rho\, d\Omega^2$$

to obtain

$$-dt^2 + \cosh^2 t(d\rho^2 + \sin^2 \rho\, d\Omega^2) \tag{3.13}$$

Note that it is not possible for a light ray to circumnavigate the closed 3-space in this model, since the null geodesics are represented by straight lines on the quadric.

The conventional (expanding, flat 3-space) de-Sitter metric (1.13) corresponds to the intersection of the quadric by the planes $\eta_4 + \eta_5 =$ constant (which are parabolae in the (r, η_4, η_5)-space) and the planes $\eta_4 + \eta_5 + Kr = 0$ (which pass through the origin and therefore yield geodesics) (Fig. 4). We define the parameters ρ and t:

$$\left.\begin{aligned} \rho &= r/(\eta_4 + \eta_5) \\ e^t &= \eta_4 + \eta_5 \end{aligned}\right\} \tag{3.14}$$

Then

$$r = \rho\, e^t$$

and

$$\eta_4 - \eta_5 = \frac{1 - r^2}{\eta_4 + \eta_5} = (1 - \rho^2 e^{2t})\, e^{-t}$$

Differentiating these expressions and forming the metric

$$dr^2 + (d\eta_5 + d\eta_4)(d\eta_5 - d\eta_4)$$

we get

$$-dt^2 + e^{2t}\, d\rho^2 \tag{3.15}$$

to which we have to add $r^2 d\Omega^2 = \rho^2 e^{2t} d\Omega^2$ to obtain just (1.13). Note that the whole 4-space corresponds to the part of the quadric above the plane $\eta_4 = r$. Note also that the null generator through (1,0,0) shown broken in Fig. 4 is an *event horizon* in the sense that a generator through any point in the region $r > 1$ will never interesect the curve $\rho = 0$ (an observer moving along $\rho = 0$ will never observe events for which $|r| > 1$). The same event horizon is also indicated in Figs. 2 and 3.

4. *The Group SO*(2, 1)

The group of rotations $SO(2,1)$ in the (r, η_4, η_5)-space (metric $(+-+)$) leaves the quadric invariant (the quadric is the analogue of a 'sphere' in the space with indefinite metric). It is illuminating to demonstrate those particular $SO(2,1)$ rotations that leave the various coordinate lines unchanged. Consider the coordinate system (Fig. 1) corresponding to the static de-Sitter metric and rotate about the r-axis:

$$\begin{pmatrix} \eta_4 \\ \eta_5 \end{pmatrix} \to \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} \eta_4 \\ \eta_5 \end{pmatrix} \tag{4.1}$$

Then

$$\tanh t = \frac{\eta_4}{\eta_5} \to \frac{(\eta_4/\eta_5) + \tanh\chi}{1 + (\eta_4/\eta_5)\tanh\chi} = \tanh(t + \chi) \tag{4.2}$$

so that the rotation (4.1) corresponds to

$$r \to r, \qquad t \to t + \chi \tag{4.2}$$

This corresponds to the fact that the metric is 'static'—there is no 'privileged' value of t (this is not apparent in Fig. 1 where $t = 0$ is a circle and other $t =$ constant curves appear as ellipses. In fact they are *all* circles; the distortion is due to the impossibility of representing the geometry of space with metric $(+-+)$ adequately in Euclidean 3-space).

Applying to Fig. 2 a rotation about the η_5-axis

$$\begin{pmatrix} r \\ \eta_4 \end{pmatrix} \to \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} r \\ \eta_4 \end{pmatrix} \tag{4.3}$$

we get

$$\tanh\rho = \frac{r}{\eta_4} \to \tanh(\rho + \chi)$$

so the rotation (4.3) has the effect

$$\rho \to \rho + \chi, \qquad t \to t \tag{4.4}$$

This illustrates the absence of a privileged $\rho =$ constant curve. For Fig. 3 we carry out the rotation about the η_4-axis

$$\begin{pmatrix} r \\ \eta_5 \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r \\ \eta_5 \end{pmatrix} \tag{4.5}$$

Thus (recalling $\tan\rho = r/\eta_5$, $\sinh t = \eta_4$),

$$\rho \to \rho + \theta, \qquad t \to t \tag{4.6}$$

The absence of a privileged $\rho =$ constant curve in Fig. 4 is demonstrated by carrying out a rotation about the common intersection of the planes

$$\eta_4 + \eta_5 + Kr = 0 \tag{4.7}$$

This transformation is of the form

$$\begin{pmatrix} r \\ \eta_4 \\ \eta_5 \end{pmatrix} \to \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1+\alpha^2/2 & \alpha^2/2 \\ -\alpha & -\alpha^2/2 & 1-\alpha^2/2 \end{pmatrix} \begin{pmatrix} r \\ \eta_4 \\ \eta_5 \end{pmatrix} \tag{4.8}$$

(It is easily verified that the matrix here belongs to $SO(2,1)$ and that it maps the set of planes (4.7) onto itself.)

We find that $\eta_4 + \eta_5$ (and therefore t) remains invariant and that

$$\rho \to \frac{r}{\eta_4 + \eta_5} \to \frac{r + \alpha(\eta_4 + \eta_5)}{\eta_4 + \eta_5} = \rho + \alpha \tag{4.9}$$

Finally, we note that the shape of the coordinate net in Fig. 4 is unchanged by a time translation combined with a dilatation of 3-space. This transformation is simply rotation about the r-axis

$$\begin{pmatrix} \eta_4 \\ \eta_5 \end{pmatrix} \to \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} \eta_4 \\ \eta_5 \end{pmatrix} \tag{4.10}$$

so that by (3.14),

$$t \to t + \chi, \qquad \rho \to e^{\chi}\rho \tag{4.11}$$

5. *The* (3 + 2) *de-Sitter Space*

We have shown that the three metric (1.12)–(1.14) correspond to different coordinate systems on the hyperquadric

$$\eta^{AB}\eta_A\eta_B = 1 = r^2 - \eta_4{}^2 + \eta_5{}^2 \tag{5.1}$$

ina 5-space with metric $\eta_{AB} = dg(+++-+)$. To deal similarly with negative Λ we make use of the hyperquadric

$$\zeta^{AB}\eta_A\eta_B = 1 = -r^2 + \eta_4{}^2 + \eta_5{}^2 \tag{5.2}$$

in a 5-space with metric $\zeta_{AB} = dg(----++)$. The corresponding quadric in (r, η_4, η_5)-space is different from the previous case in that the roles of r and η_4 are interchanged. It is immediately apparent that this quadric contains closed time-like geodesics (Fig. 5). This is usually taken to mean that negative Λ is non-physical. However, since a metric gives no information about the topological properties of the hyperquadric it is quite possible to consider it as a hypersurface with infinitely many sheets, so that in fact a

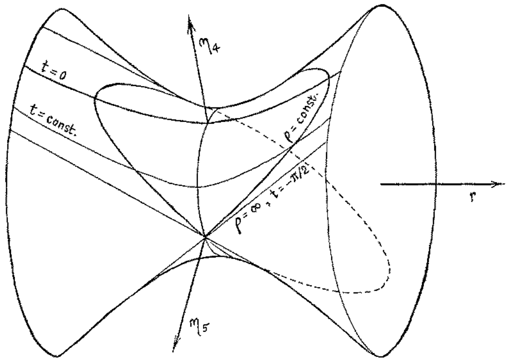

Figure 5. Universe with k = −1 and negative Λ

closed time-like line passing once round the hyperboloid would return to a *different* event having the same (η_4, η_5, r)-coordinates as its initial event.

We set up a coordinate system having the same form as that of Fig. 1, defining (ρ, t) by

$$\left.\begin{aligned} \rho &= \tanh^{-1}(r/\eta_5) \\ \eta_4 &= \sin t \end{aligned}\right\} \tag{5.3}$$

The metric for the hyperquadric becomes

$$dt^2 - \cos^2 t\, d\rho^2 \tag{5.4}$$

to which we must add

$$-r^2\, d\Omega^2 = -\cos^2 t \sinh^2 \rho\, d\Omega^2$$

to obtain (1.11). We see that the question of closed time-like lines does not arise because the universe implied by (1.11) has a beginning and an end (at $t = \pm\pi/2$) so the physically relevant portions of the time-like geodesics are only halves of the closed curves.

References

Adler, R., Bazin, M. and Schiffer, M. (1965). *Introduction to General Relativity*. McGraw-Hill.
Bondi, H. (1960). *Cosmology*, 2nd Ed. Cambridge University Press.
Fock, V. (1964). *The Theory of Space, Time and Gravitation*. Pergamon.
Hoyle, F. and Narlikar, J. V. (1963). *Proceedings of the Royal Society*, **A273**, 1.